Charge excitations in SU(n) spin chains: exact results for the $1/r^2$ model

Ronny Thomale, Dirk Schuricht[*], and Martin Greiter

Institut für Theorie der Kondensierten Materie,

Universität Karlsruhe, Postfach 6980, 76128 Karlsruhe, Germany

Abstract

We study the one- and two-holon excitations of the SU(3) Kuramoto–Yokoyama model on the level of explicit wave functions, and generalize the calculations to the case of SU(n). We obtain the exact energies and the single holon momenta, which we find fractionally spaced according to fractional statistics with statistical parameter $g = 1/n$.

PACS numbers: 75.10.Pq, 02.30.Ik, 75.10.Jm, 05.30.Pr

I. INTRODUCTION

Since its discovery in 1988 by Haldane [1] and Shastry [2], the Haldane–Shastry model (HSM) has amply contributed to our understanding of fractional quantization in one-dimensional spin chains. The model provides a framework to formulate and analyze spinons, the elementary excitations of one-dimensional spin chains, at the level of explicit wave functions [3, 4]. In particular, it was realized through this model that spinons in SU(2) spin chains obey half-Fermi statistics [5]. Kawakami [6] subsequently generalized the HSM from SU(2) spins to SU(n), a model in which the spinon excitations obey fractional statistics with statistical parameter $(1 - 1/n)$ [7, 8, 9, 10, 11, 12, 13].

The HSM was also generalized by Kuramoto and Yokoyama [14] to allow for mobile holes. The Kuramoto–Yokoyama Model (KYM) hence contains spin and charge degrees of freedom, described by spinon and holon excitations [15], which carry spin $\frac{1}{2}$ but no charge and charge $+1$ but no spin, respectively. While explicit wave functions for one-holon states in the SU(2) KYM were known for many years [16], the construction of the exact two-holon states was achieved only recently [17]. In particular, the single-holon momenta in these states were found to be shifted by a fraction of the units $2\pi/N$ appropriate for a chain with N sites, periodic boundary conditions (PBCs), and a lattice constant set to unity. This result was interpreted as a manifestation of half-Fermi and hence fractional statistics among the holon excitations [18], thus confirming a conclusion reached by Ha and Haldane [15] using the asymptotic Bethe ansatz, by Kuramoto and Kato [7, 8] from thermodynamics, and by Arikawa, Yamamoto, Saiga, and Kuramoto [19, 20] from the electron addition spectral function of the model. Like the HSM, the KYM can be generalized to spin symmetry SU(n) [6, 15].

In this article, we analyze the one-holon and two-holon excitations of the SU(n) KYM on the level of explicit wave functions. The article is organized as follows. In Section II, we investigate the case of SU(3). We first present the basic properties of the model including the ground state and the coloron excitations in the absence of holes, where the SU(3) KYM reduces to the SU(3) HSM studied previously in a similar framework [13]. We then construct the explicit one-holon and two-holon wave functions and derive the exact energies and single-holon momenta. In Section III, we generalize the results to SU(n). In particular, we review the basic properties of the ground state and the SU(n) spinon excitations before we derive

the one-holon and two-holon wave functions including their energies and momenta. In Section IV, we interpret our results in terms of free holons obeying fractional statistics with statistical parameter $g = 1/n$.

II. SU(3) KURAMOTO–YOKOYAMA MODEL

A. Hamiltonian

The SU(3) Kuramoto–Yokoyama model (KYM) [6] is most conveniently formulated by embedding the one-dimensional chain with periodic boundary conditions into the complex plane by mapping it onto the unit circle with the sites located at the complex positions $\eta_\alpha = \exp\bigl(i\frac{2\pi}{N}\alpha\bigr)$, where N denotes the number of sites and $\alpha = 1, \ldots, N$. For the SU(3) case, the sites can be either singly occupied by a fermion with SU(3) spin or empty. The Hamiltonian is given by

$$H_{\mathrm{SU(3)}} = -\frac{\pi^2}{N^2} \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} \frac{P_{\alpha\beta}}{|\eta_\alpha - \eta_\beta|^2}, \tag{1}$$

where $P_{\alpha\beta}$ exchanges the configurations on the sites η_α and η_β including a minus sign if both are fermionic. Rewriting (1) in terms of spin and fermion creation and annihilation operators yields

$$H_{\mathrm{SU(3)}} = \frac{2\pi^2}{N^2} \sum_{\alpha\neq\beta}^{N} \frac{1}{|\eta_\alpha - \eta_\beta|^2} P_{\mathrm{G}} \left[-\frac{1}{2} \sum_{\sigma=\mathrm{b,r,g}} \left(c^\dagger_{\alpha\sigma} c_{\beta\sigma} + c^\dagger_{\beta\sigma} c_{\alpha\sigma} \right) + \boldsymbol{J}_\alpha \cdot \boldsymbol{J}_\beta - \frac{n_\alpha n_\beta}{3} + n_\alpha - \frac{1}{2} \right] P_{\mathrm{G}}, \tag{2}$$

where we label the SU(3) spin or color index σ by the colors blue (b), red (r), and green (g). The Gutzwiller projector enforces at most single occupancy on all sites, and is explicitly given by

$$P_{\mathrm{G}} = \prod_{\alpha=1}^{N} (1 - n_{\alpha\mathrm{b}} n_{\alpha\mathrm{r}} - n_{\alpha\mathrm{b}} n_{\alpha\mathrm{g}} - n_{\alpha\mathrm{r}} n_{\alpha\mathrm{g}} + 2 n_{\alpha\mathrm{b}} n_{\alpha\mathrm{r}} n_{\alpha\mathrm{g}}), \tag{3}$$

where $n_\alpha = c^\dagger_{\alpha\mathrm{b}} c_{\alpha\mathrm{b}} + c^\dagger_{\alpha\mathrm{r}} c_{\alpha\mathrm{r}} + c^\dagger_{\alpha\mathrm{g}} c_{\alpha\mathrm{g}}$ is the charge occupation operator at site η_α. Furthermore, we have introduced $\boldsymbol{J}_\alpha = \frac{1}{2} \sum_{\sigma\tau} c^\dagger_{\alpha\sigma} \boldsymbol{\lambda}_{\sigma\tau} c_{\alpha\tau}$, the eight-dimensional SU(3) spin vector, where $\boldsymbol{\lambda}$ denotes the vector consisting of the eight Gell-Mann matrices (see App. A), and σ and τ are again SU(3) color indices. For all practical purposes, it is convenient to express the SU(3) spin operators in terms of colorflip operators $e^{\sigma\tau}_\alpha \equiv c^\dagger_{\alpha\sigma} c_{\alpha\tau}$. The Hamiltonian (2) then

becomes

$$H_{\mathrm{SU(3)}} = \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}P_{\mathrm{G}}\left[-\frac{1}{2}\sum_{\sigma}\left(c^{\dagger}_{\alpha\sigma}c_{\beta\sigma}+c^{\dagger}_{\beta\sigma}c_{\alpha\sigma}\right)+\frac{1}{2}\sum_{\sigma,\tau}e^{\sigma\tau}_{\alpha}e^{\tau\sigma}_{\beta}-\frac{n_\alpha n_\beta}{2}+n_\alpha-\frac{1}{2}\right]P_{\mathrm{G}}, \tag{4}$$

where the color double sum includes terms with $\sigma=\tau$.

The KYM is supersymmetric, *i.e.,* the Hamiltonian (1) commutes with the operators $J^{ab}=\sum_\alpha a^{\dagger}_{\alpha a}a_{\alpha b}$, where $a_{\alpha a}$ denotes the annihilation operator of a particle of species a (a runs over color indices as well as empty site) at site η_α. The traceless parts of the operators J^{ab} generate the Lie superalgebra su(1|3), which includes in particular the total spin operators $\boldsymbol{J}=\sum_{\alpha=1}^{N}\boldsymbol{J}_\alpha$. In addition, the KYM possesses a super-Yangian symmetry [15], which causes its amenability to rather explicit solution.

B. Vacuum state

We first review the state containing no excitations, *i.e.,* neither colorons nor holons. This vacuum state is the ground state at one third filling, where the SU(3) KYM reduces to the SU(3) HSM. The vacuum state for $N=3M$ (M integer) is constructed by Gutzwiller projection of a filled band (or Slater determinant (SD) state) containing a total of N SU(3) particles obeying Fermi statistics

$$|\Psi_0\rangle = P_{\mathrm{G}}\prod_{|q|\leq q_{\mathrm{F}}}c^{\dagger}_{q\mathrm{b}}\,c^{\dagger}_{q\mathrm{r}}\,c^{\dagger}_{q\mathrm{g}}\,|0\rangle \equiv P_{\mathrm{G}}\left|\Psi^{N}_{\mathrm{SD}}\right\rangle. \tag{5}$$

As $\left|\Psi^{N}_{\mathrm{SD}}\right\rangle$ is an SU(3) singlet by construction and P_{G} commutes with SU(3) rotations, $|\Psi_0\rangle$ is an SU(3) singlet as well.

If one interprets the state $|0_{\mathrm{g}}\rangle \equiv \prod_{\alpha=1}^{N}c^{\dagger}_{\alpha\mathrm{g}}\,|\,0\,\rangle$ as a reference state and the colorflip operators e^{bg} and e^{rg} as “particle creation operators”, the state (5) can be rewritten as [21, 22]

$$|\Psi_0\rangle = \sum_{\{z_i;w_k\}}\Psi_0[z_i;w_k]\;e^{\mathrm{bg}}_{z_1}\ldots e^{\mathrm{bg}}_{z_{M_1}}\,e^{\mathrm{rg}}_{w_1}\ldots e^{\mathrm{rg}}_{w_{M_2}}\,|0_{\mathrm{g}}\rangle, \tag{6}$$

where the sum extends over all possible ways to distribute the positions of the blue particles $z_1,\ldots,z_{M_1}$ and red particles $w_1,\ldots,w_{M_2}$ over the N sites. The vacuum state wave function is given by

$$\Psi_0[z_i;w_k] \equiv \prod_{i<j}^{M_1}(z_i-z_j)^2\prod_{k<l}^{M_2}(w_k-w_l)^2\prod_{i=1}^{M_1}\prod_{k=1}^{M_2}(z_i-w_k)\prod_{i=1}^{M_1}z_i\prod_{k=1}^{M_2}w_k \tag{7}$$

with $M_1 = M_2 = M$, its energy is

$$E_0 = -\frac{\pi^2}{36}\left(N + \frac{15}{N}\right). \tag{8}$$

The total momentum, as defined through $e^{ip} = \Psi_0[\eta_1 z_i, \eta_1 w_k]/\Psi_0[z_i, w_k]$ with $\eta_1 = \exp(i\frac{2\pi}{N})$, is $p = 0$ regardless of M. For further purposes, it is important to note that the wave function (7) can be equally expressed by any two sets of color variables, as it is shown in App. C.

C. Coloron excitations

Let $N = 3M - 1$, $M_1 = (N-2)/3$, $M_2 = (N+1)/3$. A localized coloron at site "η_γ" is constructed by annihilation of a particle with color σ from a Slater determinant state of $N+1$ fermions before Gutzwiller projection [13]:

$$\left|\Psi^{\mathrm{c}}_{\gamma\bar{\sigma}}\right\rangle = P_{\mathrm{G}} c_{\gamma\sigma} \left|\Psi^{N+1}_{\mathrm{SD}}\right\rangle, \tag{9}$$

where $\bar{\sigma}$ denotes the complementary color of the coloron. The annihilation of the fermion causes an inhomogeneity in the SU(3) spin and charge degree of freedom. The projection, however, smoothes out the inhomogeneity in the charge degrees of freedom, the coloron thus possesses color, but no charge. The wave function of a localized, *e.g.* anti-blue or yellow, coloron is given by

$$\Psi^{\mathrm{c}}_{\gamma}[z_i; w_k] = \prod_{i=1}^{M_1} (\eta_\gamma - z_i)\Psi_0[z_i; w_k], \tag{10}$$

with Ψ_0 as stated in (7). Fourier transformation yields the momentum eigenstates

$$\left|\Psi^{\mathrm{c}}_{n}\right\rangle = \frac{1}{N}\sum_{\gamma=1}^{N} (\bar{\eta}_\gamma)^n \left|\Psi^{\mathrm{c}}_{\gamma}\right\rangle, \tag{11}$$

which identically vanish unless $0 \leq n \leq M_1$. In particular, this implies that the localized one-coloron states (9) form an overcomplete set. It is hence not possible to interpret the "coordinate" η_γ literally as the position of the coloron. The momentum of (11) is

$$p^{\mathrm{c}}_n = \frac{4\pi}{3} - \frac{2\pi}{N}\left(n + \frac{1}{3}\right), \quad 0 \leq n \leq M_1. \tag{12}$$

The momentum eigenstates (11) are found to be exact energy eigenstates of the Hamiltonian (1) with energies

$$E^{\mathrm{c}}_n = E_0 + \frac{2}{9}\frac{\pi^2}{N^2} + \epsilon^{\mathrm{c}}\big(p^{\mathrm{c}}_n\big), \tag{13}$$

where the one-coloron dispersion is given by

$$\epsilon^{\mathrm{c}}(p) = \frac{3}{4}\left(\frac{\pi^2}{9} - (p-\pi)^2\right). \tag{14}$$

Colorons obey fractional statistics, the statistical parameter between color-polarized colorons is given by $g = 2/3$.

D. One-Holon excitations

1. One-holon wave functions

If we dope holes into the SU(3) spin chain, this will cause the existence of holons, the elementary charge excitations of the system. In this section, we will construct the wave functions of the one-holon states and prove by explicit calculation that these states are eigenstates of the Hamilitonian (1). For this consider a chain with $N = 3M + 1$ sites. A localized holon at lattice site η_ξ is constructed as

$$\left|\Psi_\xi^{\mathrm{ho}}\right\rangle = c_{\xi\sigma} P_{\mathrm{G}} c_{\xi\sigma}^\dagger \left|\Psi_{\mathrm{SD}}^{N-1}\right\rangle, \tag{15}$$

where the color index σ can be chosen arbitrarily. Compared to the coloron, we eliminate the inhomogeneity in color while creating an inhomogeneity in the charge distribution after Gutzwiller projection. Thus the holon has no color but charge $e > 0$ (as the charge at site η_ξ is removed). Note that the holon is constructed as apparently being strictly localized at the coordinate ξ, as states (15) on neighboring coordinates are orthogonal. In total, there are N independent states of the form (15).

Momentum eigenstates are constructed from (15) by Fourier transformation. We will show below that only $(N+5)/3$ of them are energy eigenstates, and restrict ourselves to this subset in the following. In order to describe these states by their wave functions, we take $|0_{\mathrm{g}}\rangle \equiv \prod_{\alpha=1}^{N} c_{\alpha\mathrm{g}} |0\rangle$ as reference state and and write the one-holon states as

$$\left|\Psi_m^{\mathrm{ho}}\right\rangle = \sum_{\{z_i;w_k;h\}} \Psi_m^{\mathrm{ho}}[z_i;w_k;h]\, c_{h\mathrm{g}} e_{z_1}^{\mathrm{bg}} \ldots e_{z_{M_1}}^{\mathrm{bg}} e_{w_1}^{\mathrm{rg}} \ldots e_{w_{M_2}}^{\mathrm{rg}} |0_{\mathrm{g}}\rangle, \tag{16}$$

where the sum extends over all possible ways to distribute the blue coordinates z_i, the red coordinates w_k, and the holon coordinate h over the N sites subject to the restriction $h \neq z_i, w_k$. The one-holon wave function is given by

$$\Psi_m^{\mathrm{ho}}[z_i;w_k;h] = h^m \prod_{i=1}^{M_1}(h - z_i) \prod_{k=1}^{M_2}(h - w_k) \Psi_0[z_i;w_k]. \tag{17}$$

To increase readability of the following calculations, we will keep the distinction between M_1 and M_2, although we will always set M_1, M_2 and M_3, *i.e.,* the numbers of blue, red, and green particles, to be equal to M at the end. In order for (17) to represent energy eigenstates, the integer m has to be restricted to

$$0 \le m \le M + 1 = \frac{N+2}{3}. \tag{18}$$

For other values of m, the states $\left|\Psi_m^{\text{ho}}\right\rangle$ are not eigenstates of the Hamiltonian (1), although they do not vanish identically (as the $|\Psi_n^{\text{c}}\rangle$'s do). Consequently, we are allowed to refer to the states (16) with (17) as "holons" only if $0 \le m \le M+1$.

This also implies that the states (15) do not really constitute "holons" localized in position space, but only basis states which can be used to construct holons if the momentum is chosen adequately. Since the states (15) are orthogonal for different lattice positions ξ, there are $N = 3M+1$ orthogonal position basis states $\left|\Psi_\xi^{\text{ho}}\right\rangle$. These states cannot strictly be holons, but rather constitute incoherent superpositions of holons and other states. It is hence not possible to localize a holon onto a single lattice site. The best we can do is to take a Fourier transform of the exact eigenstates $\left|\Psi_m^{\text{ho}}\right\rangle$ for $0 \le m \le M+1$ back into position space. The resulting "localized" holon states will be true holons but will not be localized strictly onto lattice sites.

The momentum of (16) is

$$p_m^{\text{ho}} = \frac{2\pi}{3} + \frac{2\pi}{N}\left(m - \frac{1}{3}\right). \tag{19}$$

The one-holon energies are derived below to be

$$E_m^{\text{ho}} = E_0 - \frac{2}{9}\frac{\pi^2}{N^2} + \epsilon^{\text{ho}}\big(p_m^{\text{ho}}\big), \tag{20}$$

where the one-holon dispersion is given by

$$\epsilon^{\text{ho}}(p) = -\frac{3}{4}\left(\frac{\pi^2}{9} - (p-\pi)^2\right), \quad \frac{2\pi}{3} \le p \le \frac{4\pi}{3}. \tag{21}$$

In the following subsection we will prove that the states (16) are energy eigenstates of the Hamiltonian (1), if (and only if) the momentum quantum number m is restricted to (18).

2. Derivation of the one-holon energies

To evaluate the action of $H_{\text{SU}(3)}$ on $\left|\Psi_m^{\text{ho}}\right\rangle$, we first replace $e_\alpha^{\text{gg}} e_\beta^{\text{gg}}$ by $(1-h_\alpha - e_\alpha^{\text{bb}} - e_\alpha^{\text{rr}})(1- h_\beta - e_\beta^{\text{bb}} - e_\beta^{\text{rr}})$, where h_α denotes the hole occupation operator $h_\alpha = 1 - n_\alpha$, and rewrite the

Hamiltonian (4) as

$$
\begin{aligned}
H_{\mathrm{SU}(3)} \;=\;& \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(e_\alpha^{\mathrm{bg}}e_\beta^{\mathrm{gb}}+e_\alpha^{\mathrm{rg}}e_\beta^{\mathrm{gr}}+e_\alpha^{\mathrm{br}}e_\beta^{\mathrm{rb}}\right)\\
&+\frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(e_\alpha^{\mathrm{bb}}e_\beta^{\mathrm{bb}}+e_\alpha^{\mathrm{rr}}e_\beta^{\mathrm{rr}}+e_\alpha^{\mathrm{bb}}e_\beta^{\mathrm{rr}}\right)\\
&-\frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(e_\alpha^{\mathrm{bb}}+e_\alpha^{\mathrm{rr}}\right)+\frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(n_\alpha-\frac{1}{2}\right)\\
&+\frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(e_\alpha^{\mathrm{bb}}+e_\alpha^{\mathrm{rr}}\right)(1-n_\beta)\\
&+\frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}\left[\frac{1}{2}(c_{\alpha\mathrm{b}}c_{\beta\mathrm{b}}^\dagger+c_{\alpha\mathrm{r}}c_{\beta\mathrm{r}}^\dagger)+\frac{1}{2}(c_{\alpha\mathrm{b}}c_{\beta\mathrm{b}}^\dagger+c_{\alpha\mathrm{g}}c_{\beta\mathrm{g}}^\dagger)\right.\\
&\qquad\qquad\left.+\frac{1}{2}(c_{\alpha\mathrm{r}}c_{\beta\mathrm{r}}^\dagger+c_{\alpha\mathrm{g}}c_{\beta\mathrm{g}}^\dagger)\right].
\end{aligned}
\tag{22}
$$

In the following we evaluate each term of (22) separately.

The first term $\left[e_\alpha^{\mathrm{bg}}e_\beta^{\mathrm{gb}}\Psi_m^{\mathrm{ho}}\right][z_i;w_k;h]$, which vanishes unless one of the z_i's is equal to η_α, yields through Taylor expansion (the derivative operators are understood to act on the analytic extension of the wave function)

$$
\left[\sum_{\alpha\neq\beta}^{N}\frac{e_\alpha^{\mathrm{bg}}e_\beta^{\mathrm{gb}}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\mathrm{ho}}\right][z_i;w_k;h]=\sum_{i=1}^{M_1}\sum_{\beta\neq i}^{N}\frac{\eta_\beta}{|z_i-\eta_\beta|^2}\frac{\Psi_m^{\mathrm{ho}}[\ldots,z_{i-1},\eta_\beta,z_{i+1},\ldots;w_k;h]}{\eta_\beta}
$$

$$
=\sum_{i=1}^{M_1}\sum_{\ell=0}^{N-1}\frac{A_\ell z_i^{\ell+1}}{\ell!}\frac{\partial^\ell}{\partial z_i^\ell}\frac{\Psi_m^{\mathrm{ho}}}{z_i} \tag{23}
$$

$$
=\frac{M_1}{12}\left(N^2+8M_1^2-6M_1(N+1)+3\right)\Psi_m^{\mathrm{ho}} \tag{24}
$$

$$
-\frac{N-3}{2}\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i}{z_i-w_k}\Psi_m^{\mathrm{ho}}+\sum_{i\neq j}^{M_1}\frac{z_i^2}{(z_i-z_j)^2}\Psi_m^{\mathrm{ho}} \tag{25}
$$

$$
+2\sum_{i\neq j}^{M_1}\sum_{k=1}^{M_2}\frac{z_i^2}{(z_i-z_j)(z_i-w_k)}\Psi_m^{\mathrm{ho}} \tag{26}
$$

$$
+\frac{1}{2}\sum_{i=1}^{M_1}\sum_{k\neq l}^{M_2}\frac{z_i^2}{(z_i-w_k)(z_i-w_l)}\Psi_m^{\mathrm{ho}} \tag{27}
$$

$$
+\sum_{i\neq j}^{M_1}\frac{2z_i^2}{(z_i-z_j)(z_i-h)}\Psi_m^{\mathrm{ho}}-\frac{N-3}{2}\sum_{i=1}^{M_1}\frac{z_i}{z_i-h}\Psi_m^{\mathrm{ho}} \tag{28}
$$

$$+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i^2}{(z_i-w_k)(z_i-h)}\Psi_m^{\mathrm{ho}}\,, \tag{29}$$

where we have used $\deg_{z_i}\Psi_m^{\mathrm{ho}}[z_i;w_k;h]=N-1$ and defined $A_\ell\equiv-\sum_{\alpha=1}^{N-1}\eta_\alpha^2(\eta_\alpha-1)^{\ell-2}$. Evaluation of the latter yields $A_0=(N-1)(N-5)/12$, $A_1=-(N-3)/2$, $A_2=1$, and $A_\ell=0$ for $2<\ell\le N-1$ (see App. D). Furthermore, we have used

$$\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-x)(y-z)}+\frac{z^2}{(z-x)(z-y)}=1,\quad x,y,z\in\mathbb{C}. \tag{30}$$

The second term $\left[e_\alpha^{\mathrm{rg}}e_\beta^{\mathrm{gr}}\Psi_m^{\mathrm{ho}}\right][z_i;w_k;h]$ can be treated in the same way, yielding together with the first term in (25)

$$-\frac{N-3}{2}\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i}{z_i-w_k}+\frac{N-3}{2}\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{w_k}{z_i-w_k}=-\frac{N-3}{2}M_1M_2.$$

One part of (26) and the term corresponding to (27) can be simplified with (30) to

$$\sum_{i\neq j}^{M_1}\sum_{k=1}^{M_2}\left(\frac{z_i^2}{(z_i-z_j)(z_i-w_k)}+\frac{1}{2}\frac{w_k^2}{(z_i-w_k)(z_j-w_k)}\right)=\frac{1}{2}M_1(M_1-1)M_2,$$

as well as similar expressions for $z_i\leftrightarrow w_k$.

The third term $\left[e_\alpha^{\mathrm{br}}e_\beta^{\mathrm{rb}}\Psi_m^{\mathrm{ho}}\right][z_i;w_k;h]$ leads to

$$\begin{aligned}
&\left[\sum_{\alpha\neq\beta}^{N}\frac{e_\alpha^{\mathrm{br}}e_\beta^{\mathrm{rb}}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\mathrm{ho}}\right][z_i;w_k;h]\\
&=\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_iw_k}{(z_i-w_k)^2}\prod_{j\neq i}^{M_1}\left(1+\frac{z_i-w_k}{z_j-z_i}\right)\prod_{l\neq k}^{M_2}\left(1-\frac{z_i-w_k}{w_l-w_k}\right)\Psi_m^{\mathrm{ho}}[z_i;w_k;h]\\
&=\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_iw_k}{(z_i-w_k)^2}\Psi_m^{\mathrm{ho}} && (31)\\
&\quad-\sum_{i\neq j}^{M_1}\sum_{k=1}^{M_2}\frac{z_iw_k}{(z_i-z_j)(z_i-w_k)}\Psi_m^{\mathrm{ho}}-\sum_{i=1}^{M_1}\sum_{k\neq l}^{M_2}\frac{z_iw_k}{(w_k-z_i)(w_k-w_l)}\Psi_m^{\mathrm{ho}} && (32)\\
&\quad+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\sum_{\mu=2}^{M_1-1}\frac{1}{\mu!}\sum_{\{a_j\}}\frac{z_iw_k(z_i-w_k)^{\mu-2}}{(z_{a_1}-z_i)\cdots(z_{a_\mu}-z_i)}\Psi_m^{\mathrm{ho}} && (33)\\
&\quad+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\sum_{\nu=2}^{M_2-1}\frac{(-1)^\nu}{\nu!}\sum_{\{b_l\}}\frac{z_iw_k(z_i-w_k)^{\nu-2}}{(w_{b_1}-w_k)\cdots(w_{b_\nu}-w_k)}\Psi_m^{\mathrm{ho}} && (34)\\
&\quad+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\sum_{\mu=1}^{M_1-1}\sum_{\nu=1}^{M_2-1}\frac{(-1)^\nu}{\mu!\nu!}\\
&\quad\times\sum_{\{a_j;b_l\}}\frac{z_iw_k(z_i-w_k)^{\mu+\nu-2}}{(z_{a_1}-z_i)\cdots(z_{a_\mu}-z_i)(w_{b_1}-w_k)\cdots(w_{b_\nu}-w_k)}\Psi_m^{\mathrm{ho}}, && (35)
\end{aligned}$$

where $\{a_j\}$ ($\{b_l\}$) is a set of integers between 1 and M_1 (M_2). The summations run over all possible ways to distribute the z_{a_j} (w_{b_l}) over the blue (red) coordinates, where z_i (w_k) is excluded. The two terms (33) and (34) vanish due to [13]

Theorem 1 *Let $M \geq 3$, $z \in \mathbb{C}$, and $z_1, \ldots, z_M \in \mathbb{C}$ distinct. Then,*

$$\sum_{i=1}^{M} \frac{z_i(z_i - z)^{M-3}}{\prod_{j\neq i}^{M}(z_j - z_i)} = 0. \tag{36}$$

The last term (35) can be simplified using a theorem due to Ha and Haldane [22]:

Theorem 2 *Let $\{a_j\}$ be a set of distinct integers between* 1 *and M_1, and $\{b_l\}$ a set of distinct integers between* 1 *and M_2. Then,*

$$\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\sum_{\mu=1}^{M_1-1}\sum_{\nu=1}^{M_2-1}\sum_{\{a_j;b_l\}} \frac{(-1)^\nu}{\mu!\nu!} \frac{z_i w_k (z_i - w_k)^{\mu+\nu-2}}{(z_{a_1} - z_i)\cdots(z_{a_\mu} - z_i)(w_{b_1} - w_k)\cdots(w_{b_\nu} - w_k)}$$
$$= -\sum_{\kappa=1}^{\min(M_1,M_2)} (M_1 - \kappa)(M_2 - \kappa).$$

Furthermore, the two terms in line (32), together with the remainder of (26) and the corresponding expression from the second term of the Hamiltonian, can be simplified to $M_1 M_2 (M_1 + M_2 - 2)\Psi_m^{\mathrm{ho}}/2$.

The 2nd and 3rd line of (22) yield

$$\begin{aligned}
&\sum_{\alpha\neq\beta}^{N} \frac{e_\alpha^{\mathrm{bb}} e_\beta^{\mathrm{bb}} + e_\alpha^{\mathrm{rr}} e_\beta^{\mathrm{rr}} + e_\alpha^{\mathrm{bb}} e_\beta^{\mathrm{rr}} - e_\alpha^{\mathrm{bb}} - e_\alpha^{\mathrm{rr}} + n_\alpha - \frac{1}{2}}{|\eta_\alpha - \eta_\beta|^2} \Psi_m^{\mathrm{ho}}[z_i; w_k; h] \\
&\quad = \frac{1}{2}\big(M_1(M_1-1) + M_2(M_2-1)\big)\Psi_m^{\mathrm{ho}} - \sum_{i\neq j}^{M_1} \frac{z_i^2}{(z_i - z_j)^2}\Psi_m^{\mathrm{ho}} - \sum_{k\neq l}^{M_2} \frac{w_k^2}{(w_k - w_l)^2}\Psi_m^{\mathrm{ho}} \\
&\qquad - \sum_{i=1}^{M_1}\sum_{k=1}^{M_2} \frac{z_i w_k}{(z_i - w_k)^2}\Psi_m^{\mathrm{ho}} - \frac{N^2 - 1}{12}\left(M_1 + M_2 - \frac{N}{2} + 1\right)\Psi_m^{\mathrm{ho}},
\end{aligned} \tag{37}$$

by which the remainder of (25), its counterpart from the second term, and (31) are cancelled.

The 4th line of (22) yields

$$\sum_{\alpha\neq\beta}^{N} \frac{\left(e_\alpha^{\mathrm{bb}} + e_\alpha^{\mathrm{rr}}\right)(1 - n_\beta)}{|\eta_\alpha - \eta_\beta|^2}\Psi_m^{\mathrm{ho}}[z_i; w_k; h] = \left(\sum_{i=1}^{M_1} \frac{1}{|z_i - h|^2} + \sum_{k=1}^{M_2} \frac{1}{|w_k - h|^2}\right)\Psi_m^{\mathrm{ho}}. \tag{38}$$

We will now evaluate the charge kinetic terms, which include the technical improvements compared to previous calculations. We will use a Taylor expansion as in (23). For the

treatment of the charge kinetic terms it is crucial that the fermionic creation and annihilation operators appearing in the expansion match with the variables of the analytically extended wave function. We wish to stress that the one-holon wave function (17) can, as the ground state wave function, be equally expressed by an arbitrary pair of sets of color variables. In (22) we have thus written the charge kinetic terms in a symmetricized way. For the first term we get

$$
\begin{aligned}
&\left[\sum_{\alpha\neq\beta}^{N}\frac{1}{2}\frac{c_{\alpha\mathrm{b}}c_{\beta\mathrm{b}}^{\dagger}+c_{\alpha\mathrm{r}}c_{\beta\mathrm{r}}^{\dagger}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\mathrm{ho}}\right][z_i;w_k;h] = \left[\sum_{\alpha\neq\beta}^{N}\frac{c_{\alpha v}c_{\beta v}^{\dagger}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\mathrm{ho}}\right][v_1,\ldots,v_{M_1+M_2};h]\\
&= \sum_{\beta\neq h}^{N}\frac{\eta_\beta^m}{|h-\eta_\beta|^2}\frac{\Psi_m^{\mathrm{ho}}[v_1,\ldots,v_{M_1+M_2};\eta_\beta]}{\eta_\beta^m}\\
&= \sum_{\ell=0}^{M_1+M_2}\sum_{\beta\neq h}^{N}\frac{\eta_\beta^m(\eta_\beta-h)^\ell}{\ell!|h-\eta_\beta|^2}\frac{\partial^\ell}{\partial\eta_\beta^\ell}\left(\frac{\Psi_m^{\mathrm{ho}}[v_1,\ldots,v_{M_1+M_2};\eta_\beta]}{\eta_\beta^m}\right)\Big|_{\eta_\beta=h}\\
&= \sum_{\ell=0}^{M_1+M_2}\frac{B_\ell^m h^{m+\ell}}{\ell!}\frac{\partial^\ell}{\partial h^\ell}\left(\frac{\Psi_m^{\mathrm{ho}}[z_1,\ldots,z_{M_1};w_1,\ldots,w_{M_2};h]}{h^m}\right)\\
&= \left[\left(\frac{N^2-1}{12}+\frac{m(m-N)}{2}\right)h^m-\left(\frac{N-1}{2}-m\right)h^{m+1}\frac{\partial}{\partial h}+\frac{1}{2}h^{m+2}\frac{\partial^2}{\partial h^2}\right]\frac{\Psi_m^{\mathrm{ho}}[z_i;w_k;h]}{h^m}\\
&= \left(\frac{N^2-1}{12}+\frac{m(m-N)}{2}\right)\Psi_m^{\mathrm{ho}}-\left(\frac{N-1}{2}-m\right)\left[\sum_{i=1}^{M_1}\frac{h}{h-z_i}+\sum_{k=1}^{M_2}\frac{h}{h-w_k}\right]\Psi_m^{\mathrm{ho}}\\
&\quad+\frac{1}{2}\left[\sum_{i\neq j}^{M_1}\frac{h^2}{(h-z_i)(h-z_j)}+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{2h^2}{(h-z_i)(h-w_k)}+\sum_{k\neq l}^{M_2}\frac{h^2}{(h-w_k)(h-w_l)}\right]\Psi_m^{\mathrm{ho}}, \quad (39)
\end{aligned}
$$

where the v_i's denote the union of the blue and the red coordinates, and we have introduced $B_\ell^m=-\sum_{\alpha=1}^{N-1}\eta_\alpha^{m+1}(\eta_\alpha-1)^{\ell-2}$. Evaluation of the latter yields $B_0^m=(N^2-1)/12+m(m-N)/2$, $B_1^m=m-(N-1)/2$, $B_2^m=1$, and $B_\ell=0$ for $3\leq\ell$ and $0\leq m\leq(N+2)/3$ (see App. D). The restriction (18) of the allowed momentum values follows from the B-series in (39), since $B_\ell\neq 0$ for $3\leq\ell$ and $(N+2)/3<m$, in which case the calculations above are not feasible anymore.

For the evaluation of the remaining two charge kinetic terms we reexpress the wave function Ψ_m^{ho} by the other pairs of sets of color variables (see App. C). We then proceed as in (39), where we replace the green variables by the blue and red ones using the identities of App. E. Doing so, we finally arrive at

$$
\left[\sum_{\alpha\neq\beta}^{N}\frac{1}{2}\frac{c_{\alpha\mathrm{b}}c_{\beta\mathrm{b}}^{\dagger}+c_{\alpha\mathrm{g}}c_{\beta\mathrm{g}}^{\dagger}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\mathrm{ho}}\right][z_i;w_k;h]
$$

$$= \left(\frac{N^2-1}{12}+\frac{m(m-N)}{2}\right)\Psi_m^{\rm ho} - \left(\frac{N-1}{2}-m\right)\left[C_1 - \sum_{k=1}^{M_2}\frac{h}{h-w_k}\right]\Psi_m^{\rm ho}$$
$$+\frac{1}{2}\left[C_1^2 - C_2 - 2C_1\sum_{k=1}^{M_2}\frac{h}{h-w_k} + 2\sum_{k=1}^{M_2}\frac{h^2}{(h-w_k)^2} + \sum_{k\neq l}^{M_2}\frac{h^2}{(h-w_k)(h-w_l)}\right]\Psi_m^{\rm ho}, \quad (40)$$

as well as

$$\left[\sum_{\alpha\neq\beta}^{N}\frac{1}{2}\frac{c_{\alpha\rm r}c_{\beta\rm r}^{\dagger}+c_{\alpha\rm g}c_{\beta\rm g}^{\dagger}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\rm ho}\right][z_i;w_k;h]$$
$$= \left(\frac{N^2-1}{12}+\frac{m(m-N)}{2}\right)\Psi_m^{\rm ho} - \left(\frac{N-1}{2}-m\right)\left[C_1 - \sum_{i=1}^{M_1}\frac{h}{h-z_i}\right]\Psi_m^{\rm ho}$$
$$+\frac{1}{2}\left[C_1^2 - C_2 - 2C_1\sum_{i=1}^{M_1}\frac{h}{h-z_i} + 2\sum_{i=1}^{M_1}\frac{h^2}{(h-z_i)^2} + \sum_{i\neq j}^{M_1}\frac{h^2}{(h-z_i)(h-z_j)}\right]\Psi_m^{\rm ho}. \quad (41)$$

In (40) and (41) we have defined the constants $C_1 = \sum_{\alpha=1}^{N-1} 1/(1-\eta_\alpha) = (N-1)/2$ and $C_2 = \sum_{\alpha=1}^{N-1} 1/(1-\eta_\alpha)^2 = -(N^2-6N+5)/12$ (see App. D).

Now there occur several simplifications. The first term in (28) together with the appropriate terms in (39) and (41) yield

$$\sum_{i\neq j}^{M_1}\frac{2z_i^2}{(z_i-z_j)(z_i-h)} + \sum_{i\neq j}^{M_1}\frac{h^2}{(h-z_i)(h-z_j)} = M_1(M_1-1), \quad (42)$$

and the similar expression for $z_i \leftrightarrow w_k$ leads to $M_2(M_2-1)$. Furthermore, (29), its counterpart from the second term, and the appropriate term in (39) result in

$$\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i^2}{(z_i-w_k)(z_i-h)} + \sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{w_k^2}{(w_k-z_i)(w_k-h)} + \sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{h^2}{(h-z_i)(h-w_k)} = M_1M_2. \quad (43)$$

Finally, the remainder of (28), the first term of (38), and the remaining off-diagonal terms of (41) yield

$$-\frac{N-3}{2}\sum_{i=1}^{M_1}\frac{z_i}{z_i-h} + \sum_{i=1}^{M_1}\frac{1}{|z_i-h|^2} - \frac{N-1}{2}\sum_{i=1}^{M_1}\frac{h}{h-z_i} + \sum_{i=1}^{M_1}\frac{h^2}{(h-z_i)^2} = -M_1\frac{N-3}{2}, \quad (44)$$

and the similar expression for $z_i \leftrightarrow w_k$.

Summing up all terms, we obtain

$$H_{\rm SU(3)}\left|\Psi_m^{\rm ho}\right\rangle = E_m^{\rm ho}\left|\Psi_m^{\rm ho}\right\rangle \quad (45)$$

with

$$E_m = -\frac{2\pi^2}{N^2}\left[\frac{1}{72}N^3 + \frac{1}{24}N - \frac{1}{18} - \frac{3}{2}m\left(m - \frac{N+2}{3}\right)\right], \tag{46}$$

where we have set $M_1 = M_2 = M = (N-1)/3$. Using (19), E_m can now be easily brought into the form (20).

E. Two-Holon excitations

1. Momentum eigenstates

We will now investigate the two-holon eigenstates. For this, let the number of sites be given by $N = 3M + 2$. The state with two localized holons is constructed as

$$\left|\Psi^{\text{ho}}_{\xi_1\xi_2}\right\rangle = c_{\xi_1\sigma}c_{\xi_2\sigma}P_{\text{G}}c^\dagger_{\xi_1\sigma}c^\dagger_{\xi_2\sigma}\left|\Psi^{N-2}_{\text{SD}}\right\rangle. \tag{47}$$

Similar to the one-holon case, these localized states (47) do not really represent "holons" localized in position space, and we can refer to true physical holons only in momentum space. The two-holon momentum eigenstates will be most easily described by their wave functions

$$\begin{aligned}\Psi^{\text{ho}}_{mn}[z_i; w_k; h_1, h_2] =& (h_1 - h_2)(h_1^m h_2^n + h_1^n h_2^m)\\ &\times \prod_{i=1}^{M_1}(h_1 - z_i)(h_2 - z_i)\prod_{k=1}^{M_2}(h_1 - w_k)(h_2 - w_k)\Psi_0[z_i; w_k],\end{aligned} \tag{48}$$

where $h_{1,2}$ denote the holon coordinates and the integers m and n are restricted to

$$0 \le n \le m \le M + 1 = \frac{N+1}{3}. \tag{49}$$

This restriction will be derived below.

The two-holon state represented by (48) is

$$\left|\Psi^{\text{ho}}_{mn}\right\rangle = \sum_{\{z_i; w_k; h_1, h_2\}} \Psi^{\text{ho}}_{mn}[z_i; w_k; h_1, h_2]\, c_{h_1\text{g}}c_{h_2\text{g}}e^{\text{bg}}_{z_1}\dots e^{\text{bg}}_{z_{M_1}}e^{\text{rg}}_{w_1}\dots e^{\text{rg}}_{w_{M_2}}\left|0_{\text{g}}\right\rangle, \tag{50}$$

where the sum contains the restriction $h_{1,2} \neq z_i, w_k$. The total momentum of the states (50) is found to be

$$p^{\text{ho}}_{mn} = \frac{4\pi}{3} + \frac{2\pi}{N}\left(m + n - \frac{1}{3}\right) \mod 2\pi. \tag{51}$$

In the following two subsections we construct the two-holon energy eigenstates starting from (47). The used strategy is similar to the construction of the two-holon states in the SU(2) KYM [17].

2. Action of $H_{SU(3)}$ on the momentum eigenstates

In order to derive the action of the Hamiltonian on the momentum eigenstates (50), we first define the auxiliary wave functions

$$\begin{aligned}\varphi_{mn}[z_i;w_k;h_1,h_2] &= h_1^m h_2^n \prod_{i=1}^{M_1}(h_1-z_i)(h_2-z_i)\prod_{k=1}^{M_2}(h_1-w_k)(h_2-w_k)\Psi_0[z_i;w_k] \\ &\equiv \psi_{h_1h_2}\Psi_0[z_i;w_k],\end{aligned} \tag{52}$$

which can be used to express the wave functions (48) as

$$\Psi^{\text{ho}}_{mn} = \varphi_{m+1,n}+\varphi_{n+1,m}-\varphi_{m,n+1}-\varphi_{n,m+1}. \tag{53}$$

In agreement to the one-holon case, we use (22) for the Hamiltonian and concentrate on the terms which differ from the ones above. The first term $\left[e^{\text{bg}}_\alpha e^{\text{gb}}_\beta\varphi_{mn}\right][z_i;w_k;h_1,h_2]$ yields

$$\left[\sum_{\alpha\neq\beta}^{N}\frac{e^{\text{bg}}_\alpha e^{\text{gb}}_\beta}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right][z_i;w_k;h_1,h_2] = \sum_{i=1}^{M_1}\sum_{\ell=0}^{N-1}\frac{A_\ell z_i^{\ell+1}}{\ell!}\frac{\partial^\ell}{\partial z_i^\ell}\frac{\varphi_{mn}}{z_i}$$

$$= \frac{M_1}{12}\left(N^2+8M_1^2-6M_1(N+1)+3\right)\varphi_{mn}$$

$$-\frac{N-3}{2}\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i}{z_i-w_k}\varphi_{mn}+\sum_{i\neq j}^{M_1}\frac{z_i^2}{(z_i-z_j)^2}\varphi_{mn} \tag{54}$$

$$+2\sum_{i\neq j}^{M_1}\sum_{k=1}^{M_2}\frac{z_i^2}{(z_i-z_j)(z_i-w_k)}\Psi^{\text{ho}}_m+\frac{1}{2}\sum_{i=1}^{M_1}\sum_{k\neq l}^{M_2}\frac{z_i^2}{(z_i-w_k)(z_i-w_l)}\varphi_{mn} \tag{55}$$

$$+\Psi_0\sum_{i=1}^{M_1}\left(\frac{1}{2}z_i^2\frac{\partial^2}{\partial z_i^2}+\sum_{j\neq i}^{M_1}\frac{2z_i^2}{z_i-z_j}\frac{\partial}{\partial z_i}-\frac{N-3}{2}z_i\frac{\partial}{\partial z_i}\right)\psi_{h_1h_2} \tag{56}$$

$$+\Psi_0\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i^2}{z_i-w_k}\frac{\partial}{\partial z_i}\psi_{h_1h_2}. \tag{57}$$

Now, the lines (54) and (55) can be treated as in the one-holon calculation, whereas the lines (56) and (57) explicitly yield using (30):

$$\sum_{i=1}^{M_1}\left(1-\frac{h_1^2}{(h_1-h_2)(h_1-z_i)}-\frac{h_2^2}{(h_2-h_1)(h_2-z_i)}\right)\varphi_{mn}$$

$$+\sum_{i\neq j}^{M_1}\left(1-\frac{h_1^2}{(h_1-z_i)(h_1-z_j)}\right)\varphi_{mn}+\sum_{i\neq j}^{M_1}\left(1-\frac{h_2^2}{(h_2-z_i)(h_2-z_j)}\right)\varphi_{mn}$$

$$-\frac{N-3}{2}\sum_{i}^{M_1}\left(1-\frac{h_1}{h_1-z_i}\right)\varphi_{mn}-\frac{N-3}{2}\sum_{i}^{M_1}\left(1-\frac{h_2}{h_2-z_i}\right)\varphi_{mn}$$

$$+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\left(1-\frac{w_k^2}{(w_k-z_i)(w_k-h_1)}-\frac{h_1^2}{(h_1-z_i)(h_1-w_k)}\right)\varphi_{mn}$$
$$+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\left(1-\frac{w_k^2}{(w_k-z_i)(w_k-h_2)}-\frac{h_2^2}{(h_2-z_i)(h_2-w_k)}\right)\varphi_{mn}. \tag{58}$$

The term $\left[e_\alpha^{\mathrm{rg}}e_\beta^{\mathrm{gr}}\varphi_{mn}\right][z_i;w_k;h_1,h_2]$ leads to the analog result with z_i and w_k interchanged. Furthermore, the terms $\left[e_\alpha^{\mathrm{br}}e_\beta^{\mathrm{rb}}\varphi_{mn}\right][z_i;w_k;h_1,h_2]$ as well as the 2nd and 3rd line of (22) are unchanged as compared to the one-holon case. The 4th line of (22) yields

$$\sum_{\alpha\neq\beta}^{N}\frac{e_\alpha^{\mathrm{bb}}+e_\alpha^{\mathrm{rr}}}{|\eta_\alpha-\eta_\beta|^2}(1-n_\beta)\,\varphi_{mn}=\left(\left\{\sum_{i=1}^{M_1}\frac{1}{|z_i-h_1|^2}+\sum_{k=1}^{M_2}\frac{1}{|w_k-h_2|^2}\right\}+\{h_1\leftrightarrow h_2\}\right)\varphi_{mn}, \tag{59}$$

where $\{h_1\leftrightarrow h_2\}$ denotes the reappearance of the preceeding terms in curly brackets with h_1 and h_2 interchanged.

For the charge kinetic terms we obtain in analogy to the one-holon states

$$\left[\sum_{\alpha\neq\beta}^{N}\frac{1}{2}\frac{c_{\alpha\mathrm{b}}c_{\beta\mathrm{b}}^\dagger+c_{\alpha\mathrm{r}}c_{\beta\mathrm{r}}^\dagger}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right][z_i;w_k;h_1,h_2]=\left[\sum_{\alpha=h_1,h_2}\sum_{\beta\neq\alpha}^{N}\frac{c_{\alpha v}c_{\beta v}^\dagger}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right][v_i;h_1,h_2]$$
$$=\sum_{\ell=0}^{M_1+M_2}\frac{B_\ell^m h_1^{m+\ell}}{\ell!}\frac{\partial^\ell}{\partial h_1^\ell}\left(\frac{\varphi_{mn}}{h_1^m}\right)+\sum_{\ell=0}^{M_1+M_2}\frac{B_\ell^n h_2^{n+\ell}}{\ell!}\frac{\partial^\ell}{\partial h_2^\ell}\left(\frac{\varphi_{mn}}{h_2^n}\right)$$
$$=\Bigg[\left(\frac{N^2-1}{6}+\frac{m(m-N)}{2}+\frac{n(n-N)}{2}\right)h_1^m h_2^n-\left(\frac{N-1}{2}-m\right)h_1^{m+1}h_2^n\frac{\partial}{\partial h_1}$$
$$-\left(\frac{N-1}{2}-n\right)h_1^m h_2^{n+1}\frac{\partial}{\partial h_2}+\frac{1}{2}h_1^{m+2}h_2^n\frac{\partial^2}{\partial h_1^2}+\frac{1}{2}h_1^m h_2^{n+2}\frac{\partial^2}{\partial h_2^2}\Bigg]\frac{\varphi_{mn}}{h_1^m h_2^n}$$
$$=\Bigg[\frac{N^2-1}{6}+\frac{m(m-N)}{2}+\frac{n(n-N)}{2}-\left(\frac{N-1}{2}-m\right)\left(\sum_{i=1}^{M_1}\frac{h_2}{h_2-z_i}+\sum_{k=1}^{M_2}\frac{h_2}{h_2-w_k}\right)$$
$$-\left(\frac{N-1}{2}-n\right)\left(\sum_{i=1}^{M_1}\frac{h_1}{h_1-z_i}+\sum_{k=1}^{M_2}\frac{h_1}{h_1-w_k}\right)$$
$$+\Bigg\{\frac{1}{2}\sum_{i\neq j}^{M_1}\frac{h_1^2}{(h_1-z_i)(h_1-z_j)}+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{h_1^2}{(h_1-z_i)(h_1-w_k)}$$
$$+\frac{1}{2}\sum_{k\neq l}^{M_2}\frac{h_1^2}{(h_1-w_k)(h_1-w_l)}\Bigg\}+\{h_1\leftrightarrow h_2\}\Bigg]\varphi_{mn}. \tag{60}$$

In this term, the restriction of the allowed momentum eigenvalues (49) follows from the B-series as in the one-holon case.

The other charge kinetic terms are treated by using the fact that the two-holon wave function can be expressed by either pairs of color variables, as is shown in App. C. The

terms involving green variables are rewritten in terms of the z_i's and w_k's by the identities given in App. E. Thus we finally deduce for the sum of the three charge kinetic terms

$$\begin{aligned}
&\left[\sum_{\alpha\neq\beta}^{N}\sum_{\sigma}\frac{c_{\sigma\alpha}c^{\dagger}_{\sigma\beta}}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right][z_i;w_k;h_1,h_2]=\Bigg[\frac{N^2-1}{2}+\frac{3}{2}m(m-N)+\frac{3}{2}n(n-N)\\
&\quad+(n+m)(N-2)-2C_1^2-2C_2-(m-n)\frac{h_1+h_2}{h_1-h_2}+\\
&\quad\Bigg\{-\frac{N-1}{2}\left(\sum_{i=1}^{M_1}\frac{h_1}{h_1-z_i}+\sum_{k=1}^{M_2}\frac{h_1}{h_1-w_k}\right)\\
&\quad+\sum_{k=1}^{M_2}\frac{h_1^2}{(h_1-w_k)^2}+\frac{h_1}{h_1-h_2}\left(\sum_{i=1}^{M_1}\frac{h_1}{h_1-z_i}+\sum_{k=1}^{M_2}\frac{h_1}{h_1-w_k}\right)\\
&\quad+\sum_{i=1}^{M_1}\frac{h_1^2}{(h_1-z_i)^2}+\sum_{i\neq j}^{M_1}\frac{h_1^2}{(h_1-z_i)(h_1-z_j)}+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{h_1^2}{(h_1-z_i)(h_1-w_k)}\\
&\quad+\sum_{k\neq l}^{M_2}\frac{h_1^2}{(h_1-w_k)(h_1-w_l)}+2\frac{h_1^2}{(h_1-h_2)^2}\Bigg\}+\{h_1\leftrightarrow h_2\}\Bigg]\varphi^{\text{ho}}_{nm},
\end{aligned}\tag{61}$$

where the constants C_1 and C_2 are defined as above.

As can be readily verified, all non-diagonal terms cancel. Summing up the diagonal contributions, we obtain the action of $H_{\text{SU(3)}}$ on the auxiliary wave functions φ_{mn},

$$\begin{aligned}
H_{\text{SU(3)}}\varphi_{mn}=&\frac{2\pi^2}{N^2}\Bigg[\frac{1}{72}(-40+33N-N^3)+\frac{3}{2}m(m-N)+\frac{3}{2}n(n-N)\\
&\qquad+(n+m)(N-2)+2\frac{h_1^2+h_2^2}{(h_1-h_2)^2}-(m-n)\frac{h_1+h_2}{h_1-h_2}\Bigg]\varphi_{mn}.
\end{aligned}\tag{62}$$

Using (53), we thus deduce

$$\begin{aligned}
H_{\text{SU(3)}}\Psi^{\text{ho}}_{mn} \;=\; &-\frac{\pi^2}{36}\left(N+\frac{3}{N}+\frac{4}{N^2}\right)\Psi^{\text{ho}}_{mn}\\
&+\frac{3\pi^2}{N^2}\left[\left(m-\frac{N+1}{3}\right)m+\left(n-\frac{N+1}{3}\right)n+\frac{m-n}{3}\right]\Psi^{\text{ho}}_{mn}\\
&+\frac{2\pi^2}{N^2}(m-n)\sum_{\ell=1}^{\lfloor\frac{m-n}{2}\rfloor}\Psi^{\text{ho}}_{m-\ell,n+\ell},
\end{aligned}\tag{63}$$

where we have used $\frac{x+y}{x-y}(x^my^n-x^ny^m)=2\sum_{l=0}^{m-n}x^{m-l}y^{n+l}-(x^my^n+x^ny^m)$ and $\lfloor\ \rfloor$ denotes the floor function, *i.e.,* $\lfloor x\rfloor$ is the largest integer $l\leq x$. First, note that the action of the Hamiltonian on Ψ^{ho}_{mn} is trigonal, *i.e.,* the "scattering" in the last line is only to smaller values of $m-n$. Second, (63) shows that the states Ψ^{ho}_{mn} form a non-orthogonal set, out of which we can construct an orthogonal basis of eigenfunctions as it is shown in the following.

3. *Energy eigenstates*

Using the Ansatz

$$\left|\Phi^{\mathrm{ho}}_{mn}\right\rangle = \sum_{\ell=0}^{\lfloor\frac{m-n}{2}\rfloor} a_\ell^{mn}\left|\Psi^{\mathrm{ho}}_{m-\ell,n+\ell}\right\rangle, \tag{64}$$

for the diagonalization of (63), we obtain the recursion relation

$$a_\ell^{mn} = -\frac{1}{3\ell\big(\ell+m-n-\frac{1}{3}\big)}\sum_{l=0}^{\ell-1}(n-m-2l)\,a_l^{mn}, \quad a_0^{mn}=1, \tag{65}$$

which defines the two-holon energy eigenstates (64). The corresponding energies are given by

$$E^{\mathrm{ho}}_{mn} = -\frac{\pi^2}{36}\left(N+\frac{3}{N}+\frac{4}{N^2}\right)+\frac{3\pi^2}{N^2}\left[\left(m-\frac{N+1}{3}\right)m+\left(n-\frac{N+1}{3}\right)n+\frac{m-n}{3}\right], \tag{66}$$

where the momentum quantum numbers are restricted to the interval (49) and the total momentum is given by (51).

The two-holon energies can be rewritten using the one-holon dispersion (21) as

$$E^{\mathrm{ho}}_{mn} = E_0 - \frac{4}{9}\frac{\pi^2}{N^2} + \epsilon^{\mathrm{ho}}\big(p^{\mathrm{ho}}_m\big)+\epsilon^{\mathrm{ho}}\big(p^{\mathrm{ho}}_n\big), \tag{67}$$

where we have introduced single-holon momenta according to

$$p^{\mathrm{ho}}_m = \frac{2\pi}{3}+\frac{2\pi}{N}m, \qquad p^{\mathrm{ho}}_n = \frac{2\pi}{3}+\frac{2\pi}{N}\left(n-\frac{1}{3}\right). \tag{68}$$

We will discuss the physical interpretation of this assignment in Section IV.

III. SU(n) KURAMOTO–YOKOYAMA MODEL

In this section we extend our investigations to the SU(n) KYM. We will concentrate on stating the results and make only short remarks on the calculation, since the decisive methods were already discussed in detail for the SU(3) case.

A. Hamiltonian

Consider an underdoped chain with at most one particle per lattice site carrying an internal SU(n) quantum number which transforms according to the fundamental representation

$\boldsymbol{n}$ of SU(n). Starting from the general expression (1) for the SU(n) KYM, the Hamiltonian can be rewritten as

$$H_{\mathrm{SU}(n)} = \frac{2\pi^2}{N^2} \sum_{\alpha\neq\beta}^{N} \frac{1}{|\eta_\alpha - \eta_\beta|^2} P_{\mathrm{G}} \left[-\frac{1}{2} \sum_{\sigma} \left(c_{\alpha\sigma}^{\dagger} c_{\beta\sigma} + c_{\beta\sigma}^{\dagger} c_{\alpha\sigma} \right) + \frac{1}{2} \sum_{\sigma,\tau} e_{\alpha}^{\sigma\tau} e_{\beta}^{\tau\sigma} - \frac{n_\alpha n_\beta}{2} + n_\alpha - \frac{1}{2} \right] P_{\mathrm{G}}, \tag{69}$$

where the summation index σ runs over all flavors $1, \ldots, n$, and the Gutzwiller projector P_{G} enforces at most single occupancy on all lattice sites. The model possesses an SU($1|n$) symmetry generated by the traceless parts of the operators $J^{ab} = \sum_\alpha a_{\alpha a}^{\dagger} a_{\alpha b}$, where $a_{\alpha a}$ annihilates a particle of flavor a at site η_α, as well as a super-Yangian symmetry [15].

B. Vacuum state

We first consider the state containing no excitations. We use a polarized state of particles of flavor n as reference state and label the coordinates of the particles of flavor $\sigma, 1 \leq \sigma \leq n-1$, by $z_i^\sigma, 1 \leq i \leq M_\sigma$. It can be shown that the states with wave functions [21]

$$\Psi_0[z_i^\sigma] = \prod_{\sigma=1}^{n-1} \prod_{i<j}^{M_\sigma} (z_i^\sigma - z_j^\sigma)^2 \prod_{\sigma<\tau}^{n-1} \prod_{i=1}^{M_\sigma} \prod_{j=1}^{M_\tau} (z_i^\sigma - z_j^\tau) \prod_{\sigma=1}^{n-1} \prod_{i=1}^{M_\sigma} z_i^\sigma \tag{70}$$

constitute exact eigenstates [22] of the Hamiltonian (69). For $N = nM$, $M_\sigma = M$, *i.e.*, at one nth filling, (70) is the ground state of (69) with energy

$$E_0 = -\frac{\pi^2}{12} \left(\frac{n-2}{n} N + \frac{2n-1}{N} \right). \tag{71}$$

The momentum is $p = (n-1)\pi M \bmod 2\pi$, *i.e.*, $p = 0$ for n odd and $p = 0$ or $p = \pi$ otherwise.

C. Spinon excitations

For $N = nM - 1$, localized SU(n) spinons are represented by the wave function [13]

$$\Psi_\gamma^{\mathrm{sp}}[z_i^\sigma] = \prod_{i=1}^{M_1} \left(\eta_\gamma - z_i^1 \right) \Psi_0[z_i^\sigma], \tag{72}$$

where $M_1 = M - 1$ and $M_2 = \ldots = M_{n-1} = M$. The spinons transform according to the representation $\bar{\boldsymbol{n}}$ under SU(n) transformations. Momentum eigenstates are constructed via

Fourier transformation, the spinon momenta are given by

$$p_\nu^{\rm sp} = \frac{n-1}{n}\pi N - \frac{2\pi}{N}\left(\nu + \frac{n-1}{2n}\right) \mod 2\pi, \tag{73}$$

where the momentum quantum number ν is restricted to $0 \le \nu \le M_1$. The momenta (73) fill the interval $[-\frac{\pi}{n}, \frac{\pi}{n}]$ for n even and M odd, or the interval $[\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n}]$ otherwise. The one-spinon energies are given by

$$E_m^{\rm sp} = E_0 + \frac{n^2-1}{12n}\frac{\pi^2}{N^2} + \epsilon^{\rm sp}(p_\nu^{\rm sp}), \tag{74}$$

with

$$\epsilon^{\rm sp}(p) = \begin{cases} \frac{n}{4}\left(\frac{\pi^2}{n^2} - p^2\right), & \text{if } n \text{ even and } M \text{ odd,} \\ \frac{n}{4}\left(\frac{\pi^2}{n^2} - (p-\pi)^2\right), & \text{otherwise.} \end{cases} \tag{75}$$

SU(n) spinons obey fractional statistics, the statistical parameter between spin-polarized spinons is given by $g = (n-1)/n$.

D. One-Holon excitations

For $N = nM + 1$ one can show by a straight-forward generalization of the SU(3) calculations that the one-holon states represented by the wave functions

$$\Psi_\mu^{\rm ho}[z^\sigma; h] = h^\mu \prod_{\sigma=1}^{n-1}\prod_{i=1}^{M_\sigma}\left(h - z_i^\sigma\right)\Psi_0[z^\sigma], \tag{76}$$

are eigenstates of the Hamiltonian (69). In (76), h denotes the holon coordinate, $M_1 = \ldots = M_{n-1} = M$, and the momentum quantum number μ is restricted to

$$0 \le \mu \le \frac{N+n-1}{n}. \tag{77}$$

The state corresponding to (69) is constructed in analogy to (16). The one-holon momenta are given by

$$p_\mu^{\rm ho} = \frac{n-1}{n}\pi N + \frac{2\pi}{N}\left(\mu - \frac{n-1}{2n}\right) \mod 2\pi, \tag{78}$$

which fill the interval $[-\frac{\pi}{n}, \frac{\pi}{n}]$ for n even and M odd, or the interval $[\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n}]$ otherwise (either n odd or M even or both). The one-holon energies are

$$E_m^{\rm ho} = E_0 - \frac{n^2-1}{12n}\frac{\pi^2}{N^2} + \epsilon^{\rm ho}(p_\mu^{\rm ho}), \tag{79}$$

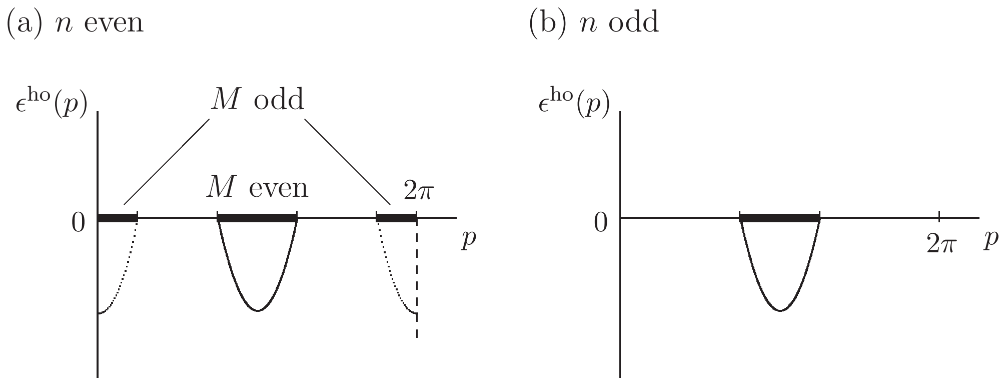

FIG. 1: SU(n) holon dispersion. a) n even. The allowed momenta fill the interval $[-\frac{\pi}{n}, \frac{\pi}{n}]$ for M odd and $[\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n}]$ for M even. b) n odd. The allowed momenta fill the interval $[\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n}]$.

with the single-holon dispersion (see Fig. 1)

$$\epsilon^{\rm ho}(p) = \begin{cases} -\frac{n}{4}\left(\frac{\pi^2}{n^2} - p^2\right), & \text{if } n \text{ even and } M \text{ odd,} \\ -\frac{n}{4}\left(\frac{\pi^2}{n^2} - (p-\pi)^2\right), & \text{otherwise.} \end{cases} \tag{80}$$

E. Two-Holon excitations

Consider a chain with $N = nM + 2$ lattice sites. The two-holon momentum eigenstates are represented by the wave function

$$\Psi^{\rm ho}_{\mu\nu}[z_\sigma; h_1, h_2] = (h_1 - h_2)(h_1^\mu h_2^\nu + h_1^\nu h_2^\mu) \prod_{\sigma=1}^{n-1} \prod_{i=1}^{M_\sigma} \big(h_1 - z_i^\sigma\big)\big(h_2 - z_i^\sigma\big)\, \Psi_0[z_i; w_k], \tag{81}$$

where the momentum quantum numbers μ and ν are restricted to

$$0 \le \nu \le \mu \le \frac{N+n-2}{n}. \tag{82}$$

The total momentum is given by

$$p^{\rm ho}_{\mu\nu} = \frac{n-1}{n}\pi N + \frac{2\pi}{N}\left(\mu + \nu - \frac{n-2}{n}\right) \mod 2\pi. \tag{83}$$

As in the SU(3) case, the momentum eigenstates (81) form a non-orthogonal basis. The two-holon energy eigenstates are obtained using the Ansatz

$$\left|\Phi^{\rm ho}_{\mu\nu}\right\rangle = \sum_{\lambda=0}^{\lfloor\frac{\mu-\nu}{2}\rfloor} a_\lambda^{\mu\nu} \left|\Psi^{\rm ho}_{\mu-\lambda,\nu+\lambda}\right\rangle, \tag{84}$$

where the recursion relation for the coefficients $a_\lambda^{\mu\nu}$'s is found to be

$$a_\lambda^{\mu\nu} = -\frac{1}{n\lambda\big(\lambda+\mu-\nu-\frac{1}{n}\big)}\sum_{\kappa=0}^{\lambda-1}(\nu-\mu-2\kappa)a_\kappa^{\mu\nu}, \quad a_0^{\mu\nu}=1. \tag{85}$$

The two-holon energies are given by

$$\begin{aligned} E_{\mu\nu}^{\rm ho} \;=\; & -\frac{\pi^2}{12n}\left(\big(n-2\big)N + \big(2n^2-13n+24\big)\frac{1}{N} - 4\big(n^2-6n+8\big)\frac{1}{N^2}\right) \\ & +\frac{n\pi^2}{N^2}\left[\left(\mu-\frac{N+n-2}{n}\right)\mu + \left(\nu-\frac{N+n-2}{n}\right)\nu + \frac{\mu-\nu}{n}\right]. \end{aligned} \tag{86}$$

Using the single-holon dispersions (80), the energy eigenvalues of (86) can be rewritten as

$$E_{\mu\nu}^{\rm ho} = E_0 - \frac{n^2-1}{6n}\frac{\pi^2}{N^2} + \epsilon^{\rm ho}\big(p_\mu^{\rm ho}\big) + \epsilon^{\rm ho}\big(p_\nu^{\rm ho}\big), \tag{87}$$

where we have introduced single-holon momenta according to

$$p_\mu^{\rm ho} = -\frac{\pi}{n} + \frac{2\pi}{N}\left(\mu - \frac{n-3}{2n}\right), \qquad p_\nu^{\rm ho} = -\frac{\pi}{n} + \frac{2\pi}{N}\left(\nu - \frac{n-1}{2n}\right), \tag{88}$$

and restricted ourselves to momenta $-\frac{\pi}{n} \le p_\nu^{\rm ho} \le p_\mu^{\rm ho} \le \frac{\pi}{n}$ for simplicity.

IV. FRACTIONAL STATISTICS

Fractional statistics in one dimension was originally introduced by Haldane [5] in terms of non-trivial state counting rules. Recently, it was realized that the fractional statistics of spinons and holons in the KYM manifests itself also in specific quantization rules for the individual spinon and holon momenta [17, 18, 23]. Here we apply this interpretation to the holon excitations of the SU(n) KYM.

First, consider holons in the SU(3) KYM. As we have seen in (67), the two-holon energies are simply given by the sum of the kinetic energies of the individual holons (and the ground state energy). This shows that the holons in the SU(3) KYM are free, which is supported by conclusions drawn from the asymptotic Bethe Ansatz [24]. Furthermore, the momentum spacing between the individual holon momenta in (68) is

$$p_m - p_n = \frac{2\pi}{N}\left(\frac{1}{3}+\ell\right), \quad \ell \in \mathbb{N}_0, \tag{89}$$

which reflects the fractional statistics of the holons with statistical parameter $g = 1/3$. This result is consistent with conclusions reached by Kuramoto and Kato [7, 8] from thermodynamics, and by Arikawa, Yamamoto, Saiga, and Kuramoto [25] from the charge dynamics of the model.

For holons in the SU(n) KYM the situation is similar. From (87) we deduce that the holons are free, whereas the momentum spacings

$$p_m - p_n = \frac{2\pi}{N}\left(\frac{1}{n} + \ell\right), \quad \ell \in \mathbb{N}_0, \tag{90}$$

obtained from (88) show that holons in the SU(n) KYM obey fractional statistics with statistical parameter $g = 1/n$.

Derived in the context of the KYM, this result has implications for SU(n) spin chains in general. In the KYM, where the holons are free in the sense that they only interact through their fractional statistics, the individual holon momenta are good quantum numbers. They assume fractionally spaced values, which for two holons are given by (90). As the statistics of the holons is a quantum invariant and as such independent of the details of the model, the fractional spacings are of universal validity as well. If we were to supplement the KYM by a potential interaction between the holons, this interaction would introduce scattering matrix elements between the exact eigenstates we obtained and labeled according to their fractionally spaced single particle momenta. These momenta would hence no longer constitute good quantum numbers. The new eigenstates would be superpositions of states with different single particle momenta, which individually, however, would still possess the fractionally shifted values. The effect of the interaction would hence be to turn the integer ℓ on the right-hand side of (90) into a superposition of integers, while leaving the fractional momentum spacing $2\pi/Nn$ unchanged.

Note that regardless of n, the sum of the statistical parameters of spinons and holons always equals the fermionic value one,

$$g_{\mathrm{sp}} + g_{\mathrm{ho}} = \frac{n-1}{n} + \frac{1}{n} = 1, \tag{91}$$

a result consistent with the concept of spin-charge separation characteristic of these models.

Finally, as models with SU(n) symmetry in general are frequently studied because of simplifying features, it is suggestive to ask whether the large-n limit deserves special attention in the model we have studied here as well. Briefly, the answer is no. No part of our calculation simplifies in this limit, as we obtain terms similar to the ones encountered above regardless of the value of n. In the limit $n \to \infty$, $g \to 0$ implies that the exclusion statistics between holons tends towards bosons. This does not mean, however, that the holons in this limit behave like free bosons, but rather that their momentum spacings shrink with the nth part of the Brillouin zone they are confined to.

V. CONCLUSIONS

In conclusion, we have constructed the explicit wave functions of the one- and two-holon excitations of the SU(n) KYM and derived their exact energies. The holons are non-interacting or free, but obey fractional statistics with parameter $g = 1/n$, which manifests itself in the quantization of the single holon momenta, which is a general feature of fractional charge excitations in SU(n) spin chains.

ACKNOWLEDGMENTS

RT was supported by a PhD scholarship of the Studienstiftung des deutschen Volkes, and DS by the Center for Functional Nanostructures (CFN) Karlsruhe.

APPENDIX A: GELL-MANN MATRICES

The Gell-Mann matrices are explicitly given by [26]

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They are normalized as $\mathrm{tr}\left(\lambda^a \lambda^b\right) = 2\delta_{ab}$ and satisfy the commutation relations $\left[\lambda^a, \lambda^b\right] = 2f^{abc}\lambda^c$. The structure constants f^{abc} are totally antisymmetric and obey Jacobi's identity

$$f^{abc}f^{cde} + f^{bdc}f^{cae} + f^{dac}f^{cbe} = 0.$$

Explicitly, the non-vanishing structure constants are given by $f^{123} = i$, $f^{147} = f^{246} = f^{257} = f^{345} = -f^{156} = -f^{367} = i/2$, $f^{458} = f^{678} = i\sqrt{3}/2$, and 45 others obtained by permutations

of the indices.

The SU(3) spin operators can be expressed in terms of the colorflip operators and the charge occupation operator as

$$\boldsymbol{J}_\alpha \cdot \boldsymbol{J}_\beta \equiv \sum_{a=1}^{8} J_\alpha^a J_\beta^a = \frac{1}{2}\sum_{\sigma\tau}^{3} e_\alpha^{\sigma\tau}\, e_\beta^{\tau\sigma} - \frac{1}{6} n_\alpha n_\beta .$$

APPENDIX B: USEFUL FORMULAS

For derivations see for example [4, 13].

1.
$$\eta_\alpha^N = 1, \quad \sum_{\alpha=1}^{N} \eta_\alpha^m = N\,\delta_{0m}, \quad \prod_{\alpha=1}^{N} \eta_\alpha = (-1)^{N-1}. \tag{B1}$$

2.
$$\frac{1}{|\eta_\alpha - \eta_\beta|^2} = -\frac{\eta_\alpha \eta_\beta}{(\eta_\alpha - \eta_\beta)^2}. \tag{B2}$$

3.
$$\prod_{\alpha=1}^{N} (\eta - \eta_\alpha) = \eta^N - 1. \tag{B3}$$

4.
$$\prod_{\beta\neq\alpha}^{N} (\eta_\beta - \eta_\alpha) = \lim_{\eta\to\eta_\alpha} \frac{\eta^N - 1}{\eta - \eta_\alpha} = \frac{N}{\eta_\alpha}. \tag{B4}$$

5.
$$\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{\eta_\alpha - 1} = \frac{N+1}{2} - m, \quad 1 \le m \le N. \tag{B5}$$

6.
$$\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{|\eta_\alpha - 1|^2} = -\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^{m+1}}{(\eta_\alpha - 1)^2} = \frac{N^2 - 1}{12} + \frac{m(m-N)}{2}, \quad 0 \le m \le N. \tag{B6}$$

APPENDIX C: REPRESENTATION OF WAVE FUNCTIONS

It is shown that the wave functions can, up to a minus sign, be expressed by any two sets of color variables. First, the wave function of the vacuum state (7) can be rewritten using

green (u) variables as

$$
\begin{aligned}
\Psi_0[z_i;w_k] &= (-1)^{M_1\frac{M_1-1}{2}}\prod_{i\neq j}^{M_1}(z_i-z_j)\prod_{k<l}^{M_2}(w_k-w_l)^2\prod_{i=1}^{M_1}\prod_{k=1}^{M_2}(z_i-w_k)\prod_{i=1}^{M_1}z_i\prod_{k=1}^{M_2}w_k \\
&= (-1)^{M_1\frac{M_1-1}{2}}(-1)^{M_1M_2}\frac{\prod_{i=1}^{M_1}z_i\prod_{k=1}^{M_2}w_k\prod_{k<l}^{M_2}(w_k-w_l)^2\prod_{i=1}^{M_1}\frac{N}{z_i}}{\prod_{i=1}^{M_1}\prod_{s=1}^{M_3}(u_s-z_i)} \\
&= (-1)^{M_1\frac{M_1-1}{2}}(-1)^{M_1M_2}\frac{\prod_{k=1}^{M_2}w_k\prod_{k<l}^{M_2}(w_k-w_l)^2N^{M_1}}{\prod_{i=1}^{M_1}\prod_{s=1}^{M_3}(u_s-z_i)},
\end{aligned}
\tag{C1}
$$

where we have used (B4). Accordingly, if we express Ψ_0 in terms of green and red variables, we find

$$
\begin{aligned}
\Psi_0[u_s;w_k] &= (-1)^{M_3\frac{M_3-1}{2}}\prod_{s\neq t}^{M_3}(u_s-u_t)\prod_{k<l}^{M_2}(w_k-w_l)^2\prod_{s=1}^{M_3}\prod_{k=1}^{M_2}(u_s-w_k)\prod_{s=1}^{M_3}u_s\prod_{k=1}^{M_2}w_k \\
&= (-1)^{M_3\frac{M_3-1}{2}}(-1)^{M_2M_3}\frac{\prod_{s=1}^{M_3}u_s\prod_{k=1}^{M_2}w_k\prod_{k<l}^{M_2}(w_k-w_l)^2\prod_{s=1}^{M_3}\frac{N}{u_s}}{\prod_{i=1}^{M_1}\prod_{s=1}^{M_3}(u_s-z_i)} \\
&= (-1)^{M_3\frac{M_3-1}{2}}(-1)^{M_2M_3}(-1)^{M_1M_3}\frac{\prod_{k=1}^{M_2}w_k\prod_{k<l}^{M_2}(w_k-w_l)^2N^{M_3}}{\prod_{i=1}^{M_1}\prod_{s=1}^{M_3}(u_s-z_i)} \\
&= (-1)^{M^2}\Psi_0[z_i;w_k],
\end{aligned}
\tag{C2}
$$

where we again used (B4), and finally set $M_1=M_2=M_3=M$.

The same line of argument can be applied to the one-holon wave functions (17),

$$
\Psi_m^{\mathrm{ho}}[z_i;w_k;h] = (-1)^{M_1\frac{M_1-1}{2}}(-1)^{M_1M_2}h^n\frac{\prod_{k=1}^{M_2}(h-w_k)\prod_{k=1}^{M_2}w_k\prod_{\substack{k,l\\k<l}}^{M_2}(w_k-w_l)^2N^{M_1}}{\prod_{i=1}^{M_1}\prod_{s=1}^{M_3}(u_s-z_i)},
\tag{C3}
$$

whereas starting with green and red variables yields

$$
\begin{aligned}
&\Psi_m^{\mathrm{ho}}[u_s;w_k;h] \\
&= (-1)^{M_3\frac{M_3-1}{2}}(-1)^{M_1M_3}(-1)^{M_2M_3}h^n\frac{\prod_{k=1}^{M_2}(h-w_k)\prod_{k=1}^{M_2}w_k\prod_{\substack{k,l\\k<l}}^{M_2}(w_k-w_l)^2N^{M_3}}{\prod_{i=1}^{M_1}\prod_{s=1}^{M_3}(u_s-z_i)} \\
&= (-1)^{M^2}\Psi_m^{\mathrm{ho}}[z_i;w_k;h].
\end{aligned}
\tag{C4}
$$

In the same way we find for the two-holon wave functions (48)

$$
\Psi_{mn}^{\mathrm{ho}}[z_i;w_k;h_1,h_2] = (-1)^{M^2}\Psi_{mn}^{\mathrm{ho}}[u_s;w_k;h_1,h_2].
\tag{C5}
$$

Thus, the holon wave functions can be expressed by any two sets of color indices. All statements generalize to SU(n).

APPENDIX D: B-SERIES

The B-series is defined as

$$B_\ell^m = -\sum_{\alpha=1}^{N-1} \eta_\alpha^{m+1}(\eta_\alpha - 1)^{\ell-2}, \tag{D1}$$

where $0 \le \ell \le 2(N-1)/3$. Now, B_0^m equals (B6), $B_1^m = m - (N-1)/2$ for $0 \le m < N$ by (B6), and $B_2^m = 1$ for $0 \le m < N+1$ by (B1). Furthermore, for $3 \le \ell \le 2(N-1)/3$ we find

$$B_\ell^m = \begin{cases} 0, & \text{for} \quad 0 \le m \le \dfrac{N+2}{3}, \\ N\dbinom{\ell-2}{N-m-1}, & \text{for} \quad \dfrac{N+2}{3} < m \le N. \end{cases} \tag{D2}$$

Proof:

$$\begin{aligned} B_\ell^m &= -\sum_{\alpha=1}^{N-1} \eta_\alpha^{m+1} \sum_{k=0}^{\ell-2} \binom{\ell-2}{k} (-1)^{\ell-k-2} \eta_\alpha^k \\ &= -\sum_{k=0}^{\ell-2} \binom{\ell-2}{k} (-1)^{\ell-k} \left(1 - \sum_{\alpha=1}^{N} \eta_\alpha^{m+k+1}\right) \\ &= -\sum_{k=0}^{\ell-2} \binom{\ell-2}{k} (-1)^{\ell-k} \big(1 - N\delta_{m,N-k-1}\big). \end{aligned}$$

Thus, for $0 \le m \le (N+2)/3$, B_ℓ^m vanishes, as the sums of the binomial coefficients of even sites and odd sites equal each other. For $(N+2)/3 < m$, however, $B_\ell^m \neq 0$, and thus the Taylor expansion appearing in the calculations of the charge kinetic terms contains higher order derivatives.

The remaining constants are deduced from $A_\ell = B_\ell^1$, $C_1 = B_1^{N-1}$, and $C_2 = -B_0^{N-1}$.

APPENDIX E: DERIVATIVE IDENTITIES

If one holon is present, we use for the simplification of the charge kinetic terms

$$\begin{aligned} \sum_{s=1}^{M_3} \frac{h}{h-u_s} &= \sum_{\alpha \neq h}^{N} \frac{h}{h-\eta_\alpha} - \sum_{i=1}^{M_1} \frac{h}{h-z_i} - \sum_{k=1}^{M_2} \frac{h}{h-w_k} \\ &= \frac{N-1}{2} - \sum_{i=1}^{M_1} \frac{h}{h-z_i} - \sum_{k=1}^{M_2} \frac{h}{h-w_k}, \end{aligned} \tag{E1}$$

as well as

$$
\begin{aligned}
\sum_{s\neq t}^{M_3}\frac{h^2}{(h-u_s)(h-u_t)} &= \sum_{s,t}^{M_3}\frac{h^2}{(h-u_s)(h-u_t)}-\sum_{s=1}^{M_3}\frac{h^2}{(h-u_s)^2}\\
&= C_1^2-C_2+\sum_{i=1}^{M_1}\frac{2h^2}{(h-z_i)^2}+\sum_{k=1}^{M_2}\frac{2h^2}{(h-w_k)^2}\\
&\quad -C_1\sum_{i=1}^{M_1}\frac{2h}{h-z_i}-C_1\sum_{k=1}^{M_2}\frac{2h}{h-w_k}+\sum_{i\neq j}^{M_1}\frac{h^2}{(h-z_i)(h-z_j)}\\
&\quad +\sum_{k\neq l}^{M_2}\frac{h^2}{(h-w_k)(h-w_l)}+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{2h^2}{(h-z_i)(h-w_k)},
\end{aligned}
\tag{E2}
$$

with the constants C_1 and C_2 as above.

For the two-holon case, we apply the identity

$$
\begin{aligned}
\sum_{s\neq t}\frac{h_1^2}{(h_1-u_s)(h_1-u_t)} &= -C_2+\sum_{i=1}^{M_1}\frac{h_1^2}{(h_1-z_i)^2}+\sum_{k=1}^{M_2}\frac{h_1^2}{(h_1-w_k)^2}+\frac{h_1^2}{(h_1-h_2)^2}\\
&\quad +\left(C_1-\sum_{i=1}^{M_1}\frac{h_1}{h_1-z_i}-\sum_{k=1}^{M_2}\frac{h_1}{h_1-w_k}-\frac{h_1}{h_1-h_2}\right)\\
&\quad \times\left(C_1-\sum_{j=1}^{M_1}\frac{h_1}{h_1-z_j}-\sum_{l=1}^{M_2}\frac{h_1}{h_1-w_l}-\frac{h_1}{h_1-h_2}\right)\\
&= -C_2+C_1^2+\sum_{i\neq j}\frac{h_1^2}{(h_1-z_i)(h_1-z_j)}+\sum_{k\neq l}\frac{h_1^2}{(h_1-w_k)(h_1-w_l)}\\
&\quad +\sum_{i=1}^{M_1}\frac{2h_1^2}{(h_1-z_i)^2}+\sum_{k=1}^{M_2}\frac{2h_1^2}{(h_1-w_k)^2}+\frac{2h_1^2}{(h_1-h_2)^2}-C_1\frac{2h_1}{h_1-h_2}\\
&\quad +\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{2h_1^2}{(h_1-z_i)(h_1-w_k)}-C_1\sum_{i=1}^{M_1}\frac{2h_1}{h_1-z_i}-C_1\sum_{k=1}^{M_2}\frac{2h_1}{h_1-w_k}\\
&\quad +\frac{2h_1}{h_1-h_2}\left(\sum_{i=1}^{M_1}\frac{h_1}{h_1-z_i}+\sum_{k=1}^{M_2}\frac{h_1}{h_1-w_k}\right),
\end{aligned}
\tag{E3}
$$

and the similar result for $h_1\leftrightarrow h_2$. All identities presented above directly generalize to SU(n).

---

[*] Present address: Rudolf Peierls Centre for Theoretical Physics, University Oxford, Oxford OX1 3NP, UK.

[1] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).

[2] B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).

[3] F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).

[4] B. A. Bernevig, D. Giuliano, and R. B. Laughlin, Phys. Rev. B **64**, 024425 (2001).

[5] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).

[6] N. Kawakami, Phys. Rev. B **46**, 1005 (1992).

[7] Y. Kuramoto and Y. Kato, J. Phys. Soc. Jpn. **64**, 4518 (1995).

[8] Y. Kato and Y. Kuramoto, J. Phys. Soc. Jpn. **65**, 1622 (1996).

[9] K. Schoutens, Phys. Rev. Lett. **79**, 2608 (1997).

[10] P. Bouwknegt and K. Schoutens, Nucl. Phys. B **547**, 501 (1999).

[11] T. Yamamoto, Y. Saiga, M. Arikawa, and Y. Kuramoto, Phys. Rev. Lett. **84**, 1308 (2000).

[12] T. Yamamoto, Y. Saiga, M. Arikawa, and Y. Kuramoto, J. Phys. Soc. Jpn. **69**, 900 (2000).

[13] D. Schuricht and M. Greiter, Phys. Rev. B. **73**, 235105 (2006).

[14] Y. Kuramoto and H. Yokoyama, Phys. Rev. Lett. **67**, 1338 (1991).

[15] Z. N. C. Ha and F. D. M. Haldane, Phys. Rev. Lett. **73**, 2887 (1994); *ibid.* **74**, E3501 (1995).

[16] R. B. Laughlin, D. Giuliano, R. Caracciolo, and O. L. White, in *Field Theories for Low-Dimensional Condensed Matter Systems*, edited by G. Morandi, P. Sodano, A. Tagliacozzo, and V. Tognetti (Springer, Berlin, 2000).

[17] R. Thomale, D. Schuricht, and M. Greiter, Phys. Rev. B **74**, 024423 (2006).

[18] M. Greiter, *Statistical Phases and Momentum Spacings for One-Dimensional Anyons*, submitted to Phys. Rev. Lett.

[19] M. Arikawa, Y. Saiga, and Y. Kuramoto, Phys. Rev. Lett. **86**, 3096 (2001).

[20] M. Arikawa, T. Yamamoto, Y. Saiga, and Y. Kuramoto, Nucl. Phys. B **702**, 380 (2004).

[21] N. Kawakami, Phys. Rev. B **46**, R3191 (1992).

[22] Z. N. C. Ha and F. D. M. Haldane, Phys. Rev. B **46**, 9359 (1992).

[23] M. Greiter and D. Schuricht, Phys. Rev. B **71**, 224424 (2005).

[24] F. H. L. Eßler, Phys. Rev. B **51**, 13357 (1995).

[25] M. Arikawa, T. Yamamoto, Y. Saiga, and Y. Kuramoto, J. Phys. Soc. Jpn. **68**, 3782 (1999).

[26] J. F. Cornwell, *Group theory in physics*, Vol. II (Academic Press, London, 1984)